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SIMILARITY SOLUTIONS IN PLANE ELASTODYNAMICS†

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Abstract—Using integral transforms, a connection is shown between Cagniard's technique and a class of similarity solutions for plane problems in elastodynamics. These solutions are given in terms of analytic functions; the functions are determined from the boundary conditions by the use of analytic function theory. This means that the techniques developed by Muskhelishvili for static elasticity may be used to solve problems within the class of similarity solutions. The Smirnov–Sobolev method is shown to be a special case of the general results derived in the paper. To illustrate the application of the general results, the half-plane stress boundary value problems of a suddenly applied line load, an expanding load, and a load over half of the bounding surface are solved in detail.

NOTATION

$D_{j\alpha}$

$$
\begin{aligned}
D_{x1}(k,p) &= -kA(k,p)\\
D_{x2}(k,p) &= -\eta_2(k)B(k,p)\\
D_{y1}(k,p) &= -\eta_1(k)A(k,p)\\
D_{y2}(k,p) &= kB(k,p)
\end{aligned}
$$

$T_{mj\alpha}$

$$
\begin{aligned}
T_{xx1}(k,p) &= (\lambda a_1^2+2\mu k^2)A(k,p)\\
T_{xx2}(k,p) &= 2\mu k\eta_2(k)B(k,p)\\
T_{yy1}(k,p) &= \mu(a_2^2-2k^2)A(k,p)\\
T_{yy2}(k,p) &= -2\mu k\eta_2(k)B(k,p)\\
T_{xy1}(k,p) &= 2\mu k\eta_1(k)A(k,p)\\
T_{xy2}(k,p) &= \mu(a_2^2-2k^2)B(k,p)\\
T_{zz1}(k,p) &= \lambda a_1^2A(k,p)\\
T_{zz2}(k,p) &= 0
\end{aligned}
$$

1. INTRODUCTION

SIMILARITY methods in applied mathematics have provided a tool for solving some problems of mathematical physics. In some cases these methods have yielded the solution to formerly unsolved problems [1, 2]; while in other cases similarity methods have proved well suited for previously solved problems (e.g., compare [3] and [4]). The derivation of the similarity form of the solution has usually been effected by defining a new set of independent variables [1–4 and references therein], and then casting the governing equations

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into a new form in terms of the new variables. The techniques employed to deduce the similarity variables range from group theory [5] to dimensional analysis [6], but, apparently, not including integral transforms. The present work uses Laplace transforms to find a general solution which, after some clearly defined assumptions, yield both the similarity variables and the similarity solution.

In a series of papers Smirnov and Sobolev [7–10] developed a similarity method for solving plane elastodynamic problems. The generality of the method noted by Miles [2] has been exploited to solve diffraction problems [11–13], crack propagation problems [3, 14, 15], and stress boundary value problems [7]. These references and their respective bibliographies show that the use of this method has been relegated almost exclusively to the Russian literature, and only mention of it has been made in other publications [2, 16, 17]. The work presented here contains the Smirnov and Sobolev method as a special case of the general theory, and sets this method on a more precise mathematical foundation. This is done, for example, by clearly stating the region of analyticity of the functions found in the theory.

2. THE GENERAL PROBLEM AND SOLUTION

Statement of the problem

In a Cartesian coordinate system, consider the half-plane $y > 0$, with bounding surface $y = 0$. Deduce the form of the response of the medium when a load is suddenly applied, at time $t = 0$, to the surface $y = 0$. The governing wave equations are

$$c_1^2\nabla^2\Phi = \frac{\partial^2\Phi}{\partial t^2}, \qquad c_2^2\nabla^2\boldsymbol{\Psi} = \frac{\partial^2\boldsymbol{\Psi}}{\partial t^2}, \qquad \nabla\,.\,\boldsymbol{\Psi} = 0. \tag{1}$$

The potentials Φ and $\boldsymbol{\Psi}$ are related to the displacements through

$$\mathbf{u} = \nabla\Phi + \nabla X\boldsymbol{\Psi}, \tag{2}$$

where ∇^2 is the Laplacian operator, c_1 and c_2 the wave speeds, $\rho c_1^2 = \lambda + 2\mu$, $\rho c_2^2 = \mu$, λ and μ are the Lamé constants, and ρ is the material density. The stress–strain relations needed in the sequel are

$$\tau_{ij} = \lambda\nabla^2\Phi\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

where δ_{ij} is the Kronecker delta. The potentials Φ and $\boldsymbol{\Psi}$, and also the displacements and stresses, are required to vanish as $y \to \infty$: that is

$$\lim_{y\to\infty}(\Phi, \boldsymbol{\Psi}, \mathbf{u}, \boldsymbol{\tau}) = 0. \tag{4}$$

The initial conditions are taken as $\Phi(x, y, 0) = \boldsymbol{\Psi}(x, y, 0) = \partial\Phi(x, y, 0)/\partial t = \partial\boldsymbol{\Psi}(x, y, 0)/\partial t = 0$, representing quiescence at $t = 0$. It will be assumed† that u_z vanishes everywhere and that u_x and u_y are independent of z. These assumptions give $\tau_{yz} = \tau_{xz} = \varepsilon_{zz} = 0$, and $\boldsymbol{\Psi} = \Psi\mathbf{e}_z$, where $\mathbf{e}_z$ is the unit vector in the z-direction.

† These assumptions yield the plane strain case [18, p. 11].

Formal solution. The one-sided and two-sided Laplace transforms to be used here are defined, respectively, by the equations

$$\bar{f}(x, y, p) = \mathscr{L}(f) = \int_0^\infty f(x, y, t)\, e^{-pt}\, \mathrm{d}t, \tag{5a}$$

$$f(x, y, t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \bar{f}(x, y, p)\, e^{pt}\, \mathrm{d}p, \tag{5b}$$

$$\bar{f}^L(k, y, p) = \int_{-\infty}^{\infty} \bar{f}(x, y, p)\, e^{pkx}\, \mathrm{d}x, \tag{6a}$$

$$\bar{f}(x, y, p) = \frac{p}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon} \bar{f}^L(k, y, p)\, e^{-pkx}\, \mathrm{d}k \tag{6b}$$

where c is chosen to the right of any singularity of $\bar{f}$. In accordance with Lerch's theorem, it is sufficient to assume in (6) that p is a real positive number [19], for this guarantees a unique inverse. In (6) $-\varepsilon$ lies within the strip of convergence [24].

The application of (5a) and (6a) to the first of equations (1), using (4) and the indicated initial conditions, leads to the ordinary differential equation for $\bar{\Phi}^L$ and its solution

$$\frac{\mathrm{d}^2\bar{\Phi}^L}{\mathrm{d}y^2} + p^2k^2\bar{\Phi}^L = a_1^2p^2\bar{\Phi}^L, \qquad a_1c_1 = 1, \tag{7}$$

$$\bar{\Phi}^L = A(k, p)\, e^{-p\eta_1(k)y}, \qquad \eta_i(k) = (a_i^2 - k^2)^{\frac{1}{2}}. \tag{8}$$

Similarly one finds that

$$\bar{\Psi}^L = B(k, p)\, e^{-p\eta_2(k)y}, \qquad a_2c_2 = 1. \tag{9}$$

The time Laplace transforms of Φ and Ψ may be written as

$$\bar{\Phi}(x, y\ p) = \frac{p}{2\pi} \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon} A(k, p)\, e^{-p[\eta_1(k)y+kx]}\, \mathrm{d}k, \tag{10}$$

$$\bar{\Psi}(x, y, p) = \frac{p}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon} B(k, p)\, e^{-p[\eta_2(k)y+kx]}\, \mathrm{d}x. \tag{11}$$

By applying (5a) and (6a) to (2), and using (8) and (9), one finds the Laplace transforms of the displacements:

$$\bar{u}_j(x, y, p) = \bar{u}_{j1}(x, y, p) + \bar{u}_{j2}(x, y, p), \qquad j = x, y, \tag{12}$$

$$\bar{u}_{j\alpha}(x, y, p) = \frac{p}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon} pD_{j\alpha}(k, p)\, e^{-p[\eta_\alpha(k)y+kx]}\, \mathrm{d}k, \qquad \alpha = 1, 2, \tag{13}$$

where the $D_{j\alpha}$ appear in the Appendix. By a similar process one deduces the Laplace transforms of the stresses as

$$\bar{\tau}_{jm}(x, y, p) = \bar{\tau}_{jm1}(x, y, p) + \bar{\tau}_{jm2}(x, y, p), \qquad j, m = x, y, \tag{14}$$

$$\bar{\tau}_{jm\alpha}(x, y, p) = \frac{p}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty-\varepsilon}^{\mathrm{i}\infty-\varepsilon} p^2 T_{mj\alpha}(k, p)\, e^{-p[\eta_\alpha(k)y+kx]}\, \mathrm{d}k. \tag{15}$$

Equations (8–15) represent the general solution of the equations of elastodynamics for the half-plane. The unknowns $A(k, p)$ and $B(k, p)$ are determined from the boundary conditions for a specific problem.

3. SIMILARITY SOLUTIONS

Using the equations of Section 2, similarity solutions will now be developed by making specific assumptions on the singularities of $A(k, p)$ and $B(k, p)$ in the complex k-plane.

A. *The similarity method*

In the evaluation of the right side of (10–15) for $x > 0$, one closes the contour to the right of the integration path as shown in Fig. 1. Paths I and II are Cagniard paths [20] along which $\mathrm{Im}(\eta_\alpha y + kx) = 0$. The similarity solution desired is obtained by assuming that

(a) the integration path of (10–15) is equivalent† to either path I or path II, depending on the ratio of x to y and the value of α, and

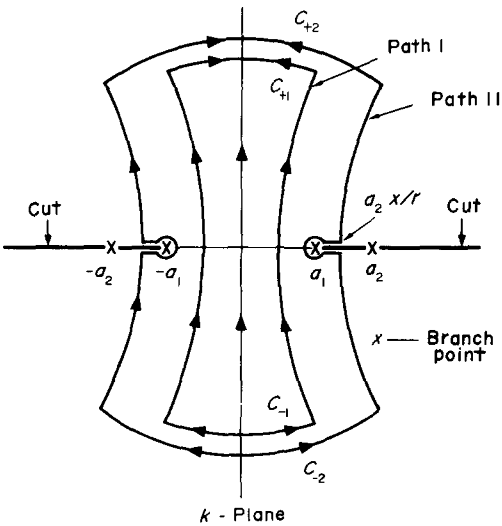

FIG. 1. Integration paths in the k-plane.

(b) $A(k, p)$ and $B(k, p)$ are separable; that is $A(k, p) = p^{-n}A(k, 1)$, $B(k, p) = p^{-n}B(k, 1)$. Under these conditions (10) reduces to

$$\bar{\Phi}(x, y, p) = \frac{p^{1-n}}{2\pi i} \int_{\text{path I}} A(k, 1)\, e^{-pt(k)}\, \mathrm{d}k, \tag{16}$$

$$t = \eta_1(k) y + kx, \tag{17}$$

† For equivalence see [21]. The basic assumptions will be considered in the last section.

where $A(k, 1)$ is an analytic function of k along the path and $\operatorname{Im} t = 0$. By changing the variable of integration in (16) from k to t one is led to

$$\begin{aligned}\frac{\partial^{n-1}\Phi(x, y, t)}{\partial t^{n-1}} &= \mathscr{L}^{-1}[p^{n-1}\bar{\Phi}(x, y, p)] = \mathscr{L}^{-1}\left[\frac{1}{2\pi i}\int_{\text{Path I}} A(k, 1)\frac{\partial k}{\partial t}e^{-pt}\,dt\right] \\ &= \mathscr{L}^{-1}\left[\frac{1}{2\pi i}\int_{a_1 r}^{\infty} A(k_1, 1)\frac{\partial k_1}{\partial t}e^{-pt}\,dt + \frac{1}{2\pi i}\int_{\infty}^{a_1 r} A(\overline{k_1}, 1)\frac{\partial \overline{k_1}}{\partial t}e^{-pt}\,dt\right] \\ &= \frac{1}{2\pi i}\left[A(k_1, 1)\frac{\partial k_1}{\partial t} - A(\overline{k_1}, 1)\frac{\partial \overline{k_1}}{\partial t}\right]\chi_1 \\ &= \operatorname{Re}\left[\frac{A(k_1, 1)}{\pi i}\cdot\frac{\partial k_1}{\partial t}\right]\chi_1 \\ &= \operatorname{Re}\frac{\partial}{\partial t}\int_0^{k_1}\frac{A(k, 1)}{\pi i}\,dk\,\chi_1, \end{aligned} \tag{18}$$

where the path of integration in the last line is confined between the origin and the Cagniard path I [$A(k, l)$ is analytic throughout the integration path], $\mathscr{L}^{-1}$ means the inverse time Laplace transform, $\overline{k_1}$ is the complex conjugate of k_1, χ_1 is the characteristic function $\chi_1 = H(t - a_1 r)$, $H(t)$ is the Heaviside unit function, $r^2 = x^2 + y^2$,

$$k_1 = \frac{\xi}{\xi^2+\eta^2} + \frac{i\eta\sqrt{[1 - a_1^2(\xi^2+\eta^2)]}}{\xi^2+\eta^2}, \qquad \xi = \frac{x}{t}, \qquad \eta = \frac{y}{t}, \tag{19}$$

and k_1 was deduced from $t - k_1 x - \eta_1(k_1)y = 0$. The integral appearing in (18) is an analytic function of k_1 [22, 23].

Restricting α to the value one ($\alpha = 1$) and following the steps shown in (18), one finds that

$$\frac{\partial^{n-2}u_{j1}(x, y, t)}{\partial t^{n-2}} = \operatorname{Re}\frac{\partial}{\partial t}\int_0^{k_1}\frac{D_{j1}(k, 1)}{\pi i}\,dk\chi_1 \tag{20}$$

$$\frac{\partial^{n-3}\tau_{jm1}(x, y, t)}{\partial t^{n-3}} = \operatorname{Re}\frac{\partial}{\partial t}\int_0^{k_1}\frac{T_{jm1}(k, 1)}{\pi i}\,dk\chi_1. \tag{21}$$

If the order of differentiation in the left side of (18), (20) and (21) is negative, then one interprets the operation as differentiation of the right side [24, Art. V.10], for example if $n = 2$ then (21) gives

$$\tau_{jm1}(x, y, t) = \operatorname{Re}\frac{\partial^2}{\partial t^2}\int_0^{k_1}\frac{T_{jm1}(k, 0, 1)}{\pi i}\,dk\chi_1. \tag{22}$$

When $\alpha = 2$ and for Ψ the steps shown in (18) must be modified to include the branch cut contributions. The final expressions are

$$\frac{\partial^{n-1}\Psi(x, y, t)}{\partial t^{n-1}} = \operatorname{Re}\sum_{l=2,3}\frac{\partial}{\partial t}\int^{k_l}\frac{B(k, 1)}{\pi i}\,dk\,\chi_l, \tag{23}$$

$$\frac{\partial^{n-2}u_{j2}(x, y, t)}{\partial t^{n-2}} = \operatorname{Re}\sum_{l=2,3}\frac{\partial}{\partial t}\int_0^{k_l}\frac{D_{j2}(k, 1)}{\pi i}\,dk\,\chi_l, \tag{24}$$

$$\frac{\partial^{n-3}\tau_{mj2}(x, y, t)}{\partial t^{n-3}} = \operatorname{Re}\sum_{l=2,3}\frac{\partial}{\partial t}\int_0^{k_l}\frac{T_{mj2}(k, 1)}{\pi i}\,dk\,\chi_l, \tag{25}$$

where χ_2 and χ_3 are the characteristic functions

$$\chi_2 = H(t-a_2 r), \tag{26}$$

$$\chi_3 = H[t-a_1 x - y(a_2^2-a_1^2)^{\frac{1}{2}}] - H(t-a_2 r), \tag{27}$$

$$k_2 = \frac{\xi}{\xi^2+\eta^2} + \frac{i\eta\sqrt{[1-a_2^2(\xi^2+\eta^2)]}}{\xi^2+\eta^2}, \tag{28}$$

$$k_3 = \frac{\xi}{\xi^2+\eta^2} - \frac{\eta\sqrt{[a_2^2(\xi^2+\eta^2)-1]}}{\xi^2+\eta^2}, \tag{29}$$

where the positive root is always taken in (28) and (29). A discussion on the determination of the proper sign for (29) is given in the Appendix.

For $x < 0$ the contour of (10)–(15) is closed to the left of the imaginary axis by the the mirror images about the origin of $C_{\pm 1}$, $C_{\pm 2}$, path I, and path II shown in Fig. 1. The desired similarity solution may be obtained by introducing a third assumption analogous to but independent of condition (a). This means that the region $x < 0$ (or $x > 0$) may admit a similarity solution independently of the form of the solution for $x > 0$ (or $x < 0$).† For $x < 0$ one assumes that condition (b) holds and that

(c) the integration path of (10)–(15) is equivalent to the mirror image of either path I or path II.

Under these assumptions it is easy to see that (18)–(22) also hold for $x < 0$. By a careful analysis one concludes that (23)–(29) hold for $x < 0$, provided that one replaces k_3 by k_4 and χ_3 by χ_4, where

$$k_4 = \frac{\xi}{\xi^2+\eta^2} + \frac{\eta\sqrt{[a_2^2(\xi^2+\eta^2)-1)]}}{\xi^2+\eta^2}$$

$$\chi_4 = H[t+a_1 x - y(a_2^2-a_1^2)^{\frac{1}{2}}] - H(t-a_2 r).$$

For example

$$\frac{\partial^{n-3}\tau_{mj2}(x,y,t)}{\partial t^{n-3}} = \mathrm{Re}\,\frac{\partial}{\partial t}\int_0^{k_2} \frac{T_{mj2}(k,1)}{\pi \mathrm{i}}\,\mathrm{d}k\,\chi_2 + \mathrm{Re}\,\frac{\partial}{\partial t}\int_0^{k_4} \frac{T_{mj2}(k,1)}{\pi \mathrm{i}}\,\mathrm{d}k\,\chi_4. \tag{25'}$$

In (18)–(25) the lower limit of integration may be changed to a point where the integrand is analytic and remains analytic throughout the integration.

B. *The Smirnov–Sobolev method*

The generality of the relations derived in the previous section allows one to deduce the equations embodied in Smirnov–Sobolev method as a special case of equations (18)–(29). These equations will now be derived to give an illustration on the use of (18)–(29) in solving plane problems. To simplify the work starred complex quantities are defined such that for all items of interest

$$f = \mathrm{Re}\{f^*\}. \tag{30}$$

† See [25] where only the region $x < 0$ admits a solution of the form given by (18)–(29). For $x > 0$ there is a pole between the imaginary axis and path I so that assumption (a) is violated.

Let us identify k_1, k_2 and k_3, respectively, with θ_1, θ_2 and θ_3 of [7]. Then, by writing $k_1 = \sigma_1 + \mathrm{i}\tau_1$ the first of equations (1) reduces to Laplace's equation

$$\frac{\partial^2 \Phi}{\partial \sigma_1^2} + \frac{\partial^2 \Phi}{\partial \tau_1^2} = 0, \tag{31}$$

corresponding to (10_1) of [7]. Similar results are found for the second of equations (1) with $k_2 = \sigma_2 + \mathrm{i}\tau_2$ and $k_3 = \sigma_3 + \tau_3$. From (31) it follows that Φ is a harmonic function and therefore can be represented by the real or imaginary part of an analytic function of k_1.

Select now $n = 2$ in (18)–(29) for $x > 0$. Then

$$\Phi^*(x, y, t) = \int_0^{k_1} \frac{A(k, 1)}{\pi \mathrm{i}} \mathrm{d}k \, \chi_1 = F(k_1)\chi_1, \tag{32}$$

$$\Psi^*(x, y, t) = \sum_{l=2,3} \int_0^{k_l} \frac{B(k, 1)}{\pi \mathrm{i}} \mathrm{d}k \, \chi_l = \sum_{l=2,3} G(k_l)\chi_l, \tag{33}$$

where $F(k_1)$, $G(k_2)$ and $G(k_3)$ are analytic functions of their respective arguments, and

$$\frac{\mathrm{d}F(k)}{\mathrm{d}k} = F'(k) = \frac{A(k, 1)}{\pi \mathrm{i}}, \qquad \frac{\mathrm{d}G(k)}{\mathrm{d}k} = G'(k) = \frac{B(k, 1)}{\pi \mathrm{i}}. \tag{34}$$

Equations (32)–(34) have the same form as the solution derived in [7]. In this reference, however, the factors χ_1, χ_2 and χ_3 were not obtained as directly as in the present work. Substituting (34) in (20), one obtains

$$\begin{aligned} u_{x1}^*(x, y, t) &= \frac{\partial}{\partial t} \int_0^{k_1} [-kF'(k)] \, \mathrm{d}k \, \chi_1 \\ &= -k_1 F'(k_1) \frac{\partial k_1}{\partial t} \chi_1. \end{aligned}$$

Using the fact that $t - xk_1 - y(a_1^2 - k_1^2)^{\frac{1}{2}} = 0$, the preceding may be written as

$$u_{x1}^*(x, y, t) = F'(k_1) \frac{\partial k_1}{\partial x} \chi_1. \tag{35}$$

Similarly the substitution of (34) into (23), using $t - xk_\alpha - y(a_2^2 - k_\alpha^2)^{\frac{1}{2}} = 0$, $\alpha = 2, 3$, leads to

$$u_x^*(x, y, t) = \chi_1 F'(k_1) \frac{\partial k_1}{\partial x} + G'(k_2) \frac{\partial k_2}{\partial y} \chi_2 + G'(k_3) \frac{\partial k_3}{\partial y} \chi_3. \tag{36}$$

One also finds that

$$u_y^*(x, y, t) = F'(k_1) \frac{\partial k_1}{\partial y} \chi_1 - G'(k_2) \frac{\partial k_2}{\partial x} \chi_2 - G'(k_3) \frac{\partial k_3}{\partial x} \chi_3. \tag{37}$$

Equations (36) and (37) may be reduced to

$$\mathbf{u}^*(x, y, t) = \nabla F(k_1)\chi_1 + \nabla x[G(k_2)\mathbf{e}_z]\chi_2 + \nabla x[G(k_3)\mathbf{e}_z]\chi_3, \tag{38}$$

in agreement with equation (20) of [7].

Proceeding now to the stresses, one finds that, for $x > 0$, (21), (25) and (34) give

$$\tau^*_{xx1}(x, y, t) = \frac{\partial^2}{\partial t^2}\int_0^{k_1} (\lambda a_1^2+2\mu k^2)F'(k)\,\mathrm{d}k\,\chi_1 \tag{39}$$

$$\tau^*_{xx2}(x, y, t) = \sum_{l=2,3} \frac{\partial^2}{\partial t^2}\int_0^{k_l} 2\mu k(a_2^2-k^2)^{\frac{1}{2}}G'(k)\,\mathrm{d}k\,\chi_l \tag{40}$$

$$\tau^*_{yy1}(x, y, t) = \frac{\partial^2}{\partial t^2}\int_0^{k_1} \mu(a_2^2-2k^2)F'(k)\,\mathrm{d}k\,\chi_1, \tag{41}$$

$$\tau^*_{yy2}(x, y, t) = -\sum_{l=2,2} \frac{\partial^2}{\partial t^2}\int_0^{k_l} 2\mu k(a_2^2-k^2)^{\frac{1}{2}}G'(k)\,\mathrm{d}k\,\chi_l, \tag{42}$$

$$\tau^*_{zz}(x, y, t) = \lambda a_1^2\frac{\partial^2 F(k_1)}{\partial t^2}\chi_1 \tag{43}$$

$$\tau^*_{yx1}(x, y, t) = \frac{\partial^2}{\partial t^2}\int_0^{k_1} 2\mu k(a_1^2-k^2)^{\frac{1}{2}}F'(k)\,\mathrm{d}k\,\chi_1 \tag{44}$$

$$\tau^*_{yx2}(x, y, t) = \sum_{l=2,3} \frac{\partial^2}{\partial t^2}\int_0^{k_l} \mu(a_2^2-2k^2)G'(k)\,\mathrm{d}k\,\chi_l \tag{45}$$

$$\tau^*_{xx}(x, y, t)+\tau^*_{yy}(x, y, t) = (\lambda a_1^2+\mu a_2^2)\frac{\partial^2 F(k_1)}{\partial t^2}\chi_1. \tag{46}$$

The corresponding equations for $x < 0$ are obtained from (32)–(46) upon replacing k_3 by k_4 and χ_3 by χ_4. If the positive square root is to be taken, then [7] fails to give the proper sign for k_4, and also the proper signs for χ_4. With these remarks, and the further condition:

(d) assumptions (a), (b) and (c) hold simultaneously,

one deduces from (30)–(46) the equations of the Smirnov–Sobolev method.

The form of the similarity solution in the Smirnov–Sobolev method restricts the imposed loads to the strip on the $y = 0$ plane satisfying $|x| < c_2 t$. In this strip $\chi_1 = \chi_2 = 1$, $\chi_3 = 0$ and $k_1 = k_2 = t/x$. Thus the boundary conditions imposed on the problem have to be such that on $y = 0$ the integrals

$$\int_0^t u(x, t)\,\mathrm{d}t, \int_0^t\int_0^{t_2} \tau(x, t_1)\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{47}$$

are functions of t/x only; here u and τ represent given displacements or stresses. It is important to note that $y = 0$ corresponds to the real axes of the complex k_1, k_2 planes, and, consequently, prescribing boundary conditions at $y = 0$ may be interpreted as prescribing Φ and Ψ, or linear functions of Φ and Ψ, along the real axes of the k_1, k_2 planes. Denoting by $f_0(t/x)$ the value of $f^*(k)$ when $\mathrm{Im}\,k = 0$, the expression for $y > 0$ may be determined from

$$f^*(k) = \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{f_0(\omega)}{\omega-k}\,\mathrm{d}\omega. \tag{48}$$

This last equation was deduced from the Poisson integral formula for the half-plane [23, p. 595]. This formula is used to find $\Phi^*(k_1)$, $\Psi^*(k_2)$, and other required functions, and all of these functions are required by the method to be analytic between the Cagniard paths of Fig. 1 and their mirror images. By comparison, the general approach given in Section 3.A requires analyticity only between the integrating path of (10)–(15) and the Cagniard paths to the right or to the left of this path.

For problems where the method has been used the reader is referred to the references in the Introduction. In the next section the method is applied for cases where n is different from 2; also problems solved using the technique of Section 3.A are given.

4. APPLICATIONS OF THE TECHNIQUE

Expanding load

Consider the half-plane described previously. A load is suddenly applied, at time $t = 0$, to the surface $y = 0$ such that

$$\tau_{yy}(x, 0, t) = \begin{cases} -I\mu & \text{for } \ |x| < t/a, \quad a_2 < a, \\ 0 & \text{otherwise}, \end{cases} \tag{49}$$

$$\tau_{xy}(x, 0, t) = 0 \quad \text{for all } x. \tag{50}$$

This load satisfies the spatial requirement for the extended Smirnov–Sobolev method, for it lies within the strip $|x| < c_2 t$ indicated before equation (47). The value of n needs to be determined.

Using (48) one finds the complex function $S^*_{yy}(x, y, t)$ such that when $y = 0$ Re $S^*_{yy}(x, 0, t) = \tau_{yy}(x, 0, t)$. This function is given by†

$$S^*_{yy}(x, y, t) = -I\mu - \frac{I\mu}{\pi \mathrm{i}} \ln \frac{k_\alpha + a}{k_\alpha - a}, \tag{51}$$

$$k_\alpha - a = l_1 \exp(i\theta_1), \qquad \theta \leq \theta_1 < 2\pi, \tag{52}$$

$$k_\alpha + a = l_2 \exp(i\theta_2), \qquad -\pi < \theta_2 \leq \pi. \tag{53}$$

(52) and (53) are dictated by the analyticity requirements of the method. The required branch cuts are shown in Fig. 2. Equations (21), (25) and (51) imply that $n = 4$. This leads to

$$-I\mu - \frac{I\mu}{\pi \mathrm{i}} \ln \frac{\omega + a}{\omega - a} = \mu \int_0^\omega [(a_2^2 - 2k^2)F'(k) - 2k(a_2^2 - k^2)^{\frac{1}{2}} G'(k)] \, \mathrm{d}k \tag{54}$$

$$0 = \mu \int_0^\omega [2k(a_1^2 - k^2)^{\frac{1}{2}} F'(k) + (a_2^2 - 2k^2) G'(k)] \, \mathrm{d}k, \tag{55}$$

where ω is the common value of k_1 and k_2 attained when $y = 0$, $\omega = t/x$. Solving (54) and (55) for F' and G', one obtains

$$F'(\omega) = -\frac{I}{\pi \mathrm{i}} \cdot \frac{(a_2^2 - 2\omega^2)}{D(\omega)} \cdot \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\ln \frac{\omega + a}{\omega - a}\right], \tag{56}$$

$$G'(\omega) = \frac{I}{\pi \mathrm{i}} \cdot \frac{2\omega(a_1^2 - \omega^2)^{\frac{1}{2}}}{D(\omega)} \cdot \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\ln \frac{\omega + a}{\omega - a}\right], \tag{57}$$

$$D(\omega) = (a_2^2 - 2\omega^2)^2 + 4\omega(a_1^2 - \omega^2)^{\frac{1}{2}}(a_2^2 - \omega^2)^{\frac{1}{2}}, \tag{58}$$

† k_α is used here so that $k_\alpha = t/x$ when $y = 0$.

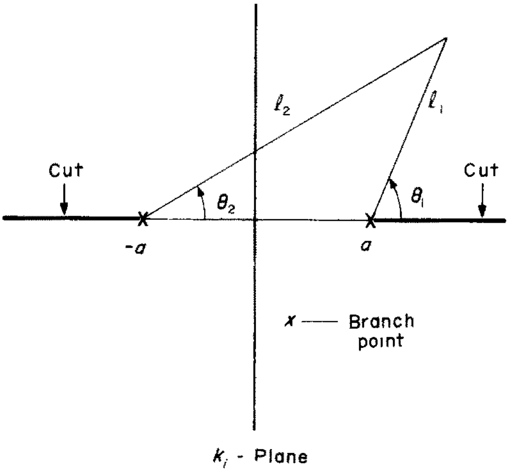

FIG. 2. Branch cuts in the k_i-plane.

and $D(\omega)$ is identified as the Rayleigh frequency equation. The only singularities of $F'(k)$ and $G'(k)$ are branch points at $\pm a_1$, $\pm a_2$, and simple poles at $\pm a$, $\pm 1/c_R$, where c_R is the Rayleigh speed. It follows that Assumption (d) is satisfied.† The stresses and accelerations are found by substituting (56)–(58) into (20)–(25), with $n = 4$.

The technique used here to derive the similarity solution yields an alternative formula for finding S^*_{yy} and S^*_{xy} which is useful even when Assumption (d) is not satisfied. This formula is given by

$$S^*_{jm}(x, y, t) = \frac{1}{\pi i}\int_0^{k_\alpha} \tilde{\tau}^L_{jm}(k, 0, 1)\,dk, \tag{59}$$

$$\begin{aligned}\tilde{\tau}^L_{jm}(k, 0, p) &= \int_{-\infty}^{\infty}\left[\int_0^{\infty} \tau_{jm}(x, 0, t)\,e^{-pt}\,dt\right] e^{pkx}\,dx,\\ &= \frac{1}{p^\sigma}\tilde{\tau}^L_{jm}(k, 0, 1),\end{aligned} \tag{60}$$

and $n = 2+\sigma$.

Uniform impulsive load for $x > 0$

When Assumption (d) is not valid, but (b) and either (a) or (c) are valid, then it is possible to use the results of Section 3.A to obtain the required solution. The procedure will be illustrated by solving the case when

$$\tau_{yy}(x, 0, t) = -\delta(t)H(x), \tag{61}$$

$$\tau_{xy}(x, 0, t) = 0. \tag{62}$$

By formula (60), one finds that

$$\tilde{\tau}^L_{yy}(k, 0, p) = \frac{1}{pk}, \qquad \operatorname{Re} k < 0. \tag{63}$$

† For stress boundary value problems the singularities of S^*_{yy} and S^*_{xy} determine whether or not Assumption (d) is satisfied.

From this it follows that $\sigma = 1$, $n = 3$, and by the remark following equation (25′),

$$S^*_{yy}(x, y, t) = \frac{1}{\pi i}\int_{-\varepsilon}^{k_\alpha} \frac{dk}{k}, \qquad \operatorname{Re} k_i < 0. \tag{64}$$

For this equation, the right branch shown in Fig. 2 with $a = 0$ is applicable. One can see that Assumption (a) is violated, but Assumption (c) still remains valid. Now that the value of n has been determined, one uses the integrals of (21) and (25′) to determine $A(k, 1)$, $B(k, 1)$, and the solution of the problem for $x < 0$:

$$\frac{1}{\pi \mathrm{i}}\int_{-\varepsilon}^{\omega} \frac{dk}{k} = \frac{1}{\pi \mathrm{i}}\int_{-\varepsilon}^{\omega} [T_{yy1}(k, 1) + T_{yy2}(k, 1)]\, \mathrm{d}k \tag{65}$$

$$0 = \int_{-\varepsilon}^{\omega} [T_{xy1}(k, 1) + T_{xy2}(k, 1)]\, \mathrm{d}k. \tag{66}$$

From these equations one finds that

$$A(k, 1) = \frac{(a_2^2 - 2k^2)}{\mu k D(k)}, \qquad B(k, 1) = -\frac{2(a_1^2 - k^2)^{\frac{1}{2}}}{\mu D(k)}. \tag{67}$$

These results agree with the velocities computed in [25] for $x < 0$. For example

$$\frac{\partial u_{x1}(x, y, t)}{\partial t} = \operatorname{Re}\frac{\partial}{\partial t}\int_{-\varepsilon}^{k_1} \frac{(2k^2 - a_2^2)}{\mu\pi \mathrm{i} D(k)}\, \mathrm{d}k\, \chi_1. \tag{68}$$

Uniform line load

This problem is known as Lamb's problem and was solved in [7] by a procedure more involved than that presented here. For this problem the boundary conditions are

$$\tau_{yy}(x, 0, t) = -\delta(t)\delta(x), \tag{69}$$

$$\tau_{xy}(x, 0, t) = 0. \tag{70}$$

This immediately leads to

$$\bar{\tau}^L_{yy}(k, 0, p) = -1, \qquad -\infty < k < \infty, \tag{71}$$

$$S^*_{yy}(x, y, t) = -\frac{1}{\pi \mathrm{i}}\int_0^{k_\alpha} \mathrm{d}k = -\frac{k_\alpha}{\pi \mathrm{i}}; \tag{72}$$

and hence $\sigma = 0$, $n = 2$. S^*_{yy} and S^*_{xy} have no singularities between the integration path of (10)–(15) and any of the Cagniard paths and therefore Assumption (d) is satisfied. F' and G' may be found by using (54) and (55), with the left side of (54) replaced by $-(\omega/\pi \mathrm{i})$. One finds that†

$$F'(\omega) = \frac{(2\omega^2 - a_2^2)}{\pi \mathrm{i} D(\omega)}, \qquad G'(\omega) = \frac{2\omega(a_1^2 - \omega^2)^{\frac{1}{2}}}{\pi \mathrm{i} D(\omega)}, \tag{73}$$

in agreement with equation (26) of [7].

† For the case $\tau_{yy}(x, 0, t) = -H(t)\delta(x)$, one simply requires $n = 3$ and uses (73) unchanged to find the solution.

Uniform step load for $x > 0$

If, instead of equation (61), one has

$$\tau_{yy}(x, 0, t) = -H(t)H(x), \tag{74}$$

then equations (49)–(58) may be used to find the solution. This is accomplished by noting that the singularities of (51) contain information about the load. For example, the branch point at $-a$ arises from the portion of the load where x is negative. $\tau_{yy}(x, 0, t)$ can be considered as the superposition of two expanding loads; that is, for $t > 0$,

$$\tau_{yy}(x, 0, t) = \tau_{yy^+}(x, 0, t) + \tau_{yy^-}(x, 0, t), \tag{75}$$

$$\tau_{yy^+}(x, 0, t) = \begin{cases} I\mu & \text{for } 0 < x < t/a \\ 0 & \text{otherwise,} \end{cases} \tag{76}$$

$$\tau_{yy^-}(x, 0, t) = \begin{cases} I\mu & \text{for } 0 < -x < t/a \\ 0 & \text{otherwise,} \end{cases} \tag{77}$$

$$S^*_{yy^+}(x, y, t) = -I\mu + \frac{I\mu}{\pi \mathrm{i}} \ln(k_\alpha - a) \tag{78}$$

$$S^*_{yy^-}(x, y, t) = -\frac{I\mu}{\pi \mathrm{i}} \ln(k_\alpha + a). \tag{79}$$

With this separation one now decomposes F' and G' into $F'_+ + F'_-$ and $G'_+ + G'_-$, respectively. In the limit $a \to 0$, F'_+ and G'_+ give the solution valid for $x < 0$ for the load of (74). This is so because as $a \to 0$ the right branch point shown in Fig. 2 approaches the origin, thus violating Assumption (a). Assumption (c) still remains valid, and, by the remark after equation (25′), the lower limit of integration is now selected as $-\varepsilon$. The required expressions for the problem are

$$F'_+(\omega) = \frac{I(a_2^2 - 2\omega^2)}{\pi \mathrm{i} D(\omega)} \frac{\mathrm{d}}{\mathrm{d}\omega}[\ln(\omega - a)], \tag{80}$$

$$G'_+(\omega) = -\frac{2I\omega(a_1^2 - \omega^2)^{\frac{1}{2}}}{\pi \mathrm{i} D(\omega)} \frac{\mathrm{d}}{\mathrm{d}\omega}[\ln(\omega - a)], \tag{81}$$

$$\lim_{a \to 0^+} F'_+(\omega) = \frac{I(a_2^2 - 2\omega^2)}{\pi \mathrm{i}\omega D(\omega)}, \tag{82}$$

$$\lim_{a \to 0^+} G'_+(\omega) = -\frac{2I(a_1^2 - \omega^2)^{\frac{1}{2}}}{\pi \mathrm{i} D(\omega)}. \tag{83}$$

Equations (78) and (79) provide a separation of $S^*_{yy}(x, y, t)$ in equation (51) of the type employed in the Wiener–Hopf technique. S_{yy^+} is analytic to the left of $k_\mathrm{i} = a$ in the k_i-plane, while S_{yy^-} is analytic to the right of $k_\mathrm{i} = -a$ in the k_i-plane. It is decompositions of this type which make the technique applicable to diffraction problems.

5. DISCUSSION OF RESULTS

In the preceding sections similarity solutions were derived for plane problems in elastodynamics. The solutions were found in terms of analytic functions of a complex variable determined from the boundary conditions. This is an important finding which permits the extension to elastodynamics of the techniques in [26, 27], as will be shown by the author in a paper presently in preparation. Under Assumption (d) the general technique reduces to the Smirnov–Sobolev method. The general technique, however, indicates the region of analyticity of the functions in the Smirnov–Sobolev method and yields the characteristic functions χ_i. Also, the similarity variables k_α are obtained in a manner simpler than that of [7].

The sample problems presented in the paper were selected only as simple illustrations of the technique, and, with the exception of the response to the uniform expanding load, the response to the other loads appears in previous works [7, 17, 20, 25]. However, in the present work the response is easily deduced from the general formulae, obviating the work in these references. The decomposition given in (75)–(83) is of the Wiener–Hopf type suitable for diffraction and crack propagation problems.

The assumptions required in deriving the general formulae (18)–(29) may be related to the applied loads and to the physical space (x, y, t), with t as a positive parameter. For assumptions (a) and (c), the easiest way to establish this relationship is to consider $t = \eta_\alpha(k_\alpha)y + k_\alpha y$ as a mapping for $y \geq 0$. One finds that† in the k_α-plane $y > 0$ maps onto $\operatorname{Im} k_\alpha > 0$; also, that $y = 0$ maps onto the real k_α-axis. The part of the boundary given by $y = 0$, $0 < x < tc_\alpha$ maps onto the semi-infinite line $\operatorname{Im} k_\alpha = 0$, $\operatorname{Re} k_\alpha > a_\alpha$, and $y = 0$, $0 > x > -tc_\alpha$ maps onto the semi-infinite line $\operatorname{Im} k_\alpha = 0$, $\operatorname{Re} k_\alpha < -a_\alpha$. By these mapping arguments, and the branch cuts which arise in the technique, one may formally conclude that Assumption (a) is satisfied for loads applied at $y = 0$ and $-\infty < x < c_2 t$; likewise one concludes that Assumption (c) is satisfied for loads applied at $y = 0$ and $-c_2 t < x < \infty$. By the notation given for $D_{j\alpha}$ and $T_{mj\alpha}$, it is easy to see that the requirement that $A(k, p)$ and $B(k, p)$ be separable (Assumption b) is equivalent to the requirement that the double transforms of the prescribed stresses or displacements be separable.

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† See also [7].

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APPENDIX. DETERMINATION OF k_3 AND k_4

The similarity variable k_3 is obtained from k_2 when the expression under the square root sign becomes negative; that is, when $1-a_2^2(\xi^2+\eta^2) < 0$. There results an ambiguity as to the sign required in k_3 so that the positive square root always be taken. This ambiguity is resolved by noting that k_3 is applicable only in region A shown in Fig. 3, and arises from the branch cut contribution. This region is bounded by the curves $t = a_2 r$ and $t = a_1 x + y(a_2^2-a_1^2)^{\frac{1}{2}}$ and lies to the right of the line $x(a_2^2-a_1^2)^{\frac{1}{2}} - a_1 y = 0$, where the inequality $x(a_2^2-a_1^2)^{\frac{1}{2}} - a_1 y > 0$ is satisfied. From $t = k_3 x + y(a_2^2-k_3^2)^{\frac{1}{2}}$ and Fig. 1, along the right branch cut k_3 increases from a_1 to $a_2 x/r$ and t increases from $t = a_1 x + y(a_2^2-a_1^2)^{\frac{1}{2}}$ to $t = a_2 r$. Thus the partial derivative of k with respect to t must be positive. This is

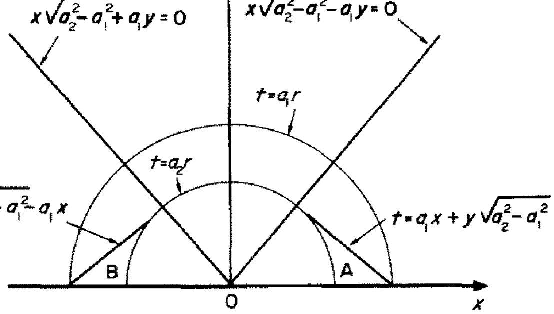

FIG. 3. Diagram for determining k_3 and k_4.

satisfied by (35), for, by using x, y and t, instead of ξ and η, one has

$$k_3 = \frac{tx}{r^2} - \frac{y\sqrt{(a_2^2 r^2 - t^2)}}{r^2}, \tag{2.1}$$

$$\frac{\partial k_3}{\partial t} = \frac{x}{r^2} + \frac{yt}{r^2\sqrt{(a_2^2 r^2 - t^2)}} > 0. \tag{2.2}$$

One further check is provided by substituting $t = a_1 x + y(a_2^2 - a_1^2)^{\frac{1}{2}}$ in (2.1). This substitution must yield $k_3 = a_1$. The substitution yields

$$k_3 = \frac{x}{r^2}\{a_1 x + y(a_2^2 - a_1^2)^{\frac{1}{2}}\} - \frac{y}{r^2}\{[x(a_2^2 - a_1^2)^{\frac{1}{2}} - ya]^2\}^{\frac{1}{2}}. \tag{2.3}$$

In region A the quantity inside the square brackets is positive. Thus since the positive square root should be taken, one obtains

$$k_3 = \frac{x}{r^2}\{a_1 x + y(a_2^2 - a_1^2)^{\frac{1}{2}}\} - \frac{y}{r^2}[x(a_2^2 - a_1^2)^{\frac{1}{2}} - ya_1] = a_1. \tag{2.4}$$

k_4 is applicable in region B of Fig. 3. This region lies to the left of the line $x(a_2^2 - a_1^2)^{\frac{1}{2}} + a_1 y = 0$, where the inequality $x(a_2^2 - a_1^2)^{\frac{1}{2}} + a_1 y < 0$ is satisfied. Along the left branch cut of Fig. 1 k_4 increases from $a_2 x/r$ to $-a_1$ and t decreases from $a_2 r$ to $y(a_2^2 - a_1^2) - a_1 x$. Thus the partial derivative of k with respect to t must be negative. This is satisfied by taking

$$k_4 = \frac{tx}{r^2} + \frac{y\sqrt{(a_2^2 r^2 - t^2)}}{r^2}, \tag{2.5}$$

and recalling that x is negative in region B. A check is provided by the substitution $t = y(a_2^2 - a_1^2)^{\frac{1}{2}} - a_1 x$, which yields

$$k_4 = \frac{x}{r^2}\{y(a_2^2 - a_1^2)^{\frac{1}{2}} - a_1 x\} + \frac{y}{r^2}\{[x(a_2^2 - a_1^2)^{\frac{1}{2}} + ya_1]^2\}^{\frac{1}{2}}. \tag{2.6}$$

In region B the quantity inside the square brackets is negative. Thus, for the positive square root, one obtains

$$k_4 = \frac{x}{r^2}\{y(a_2^2 - a_1^2)^{\frac{1}{2}} - a_1 x\} + \frac{y}{r^2}\{-x(a_2^2 - a_1^2)^{\frac{1}{2}} - ya_1\} = -a_1. \tag{2.7}$$

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Абстракт—Используя интегральные преобразования, доказывается связь между способом Каньярда и классом решений подхода для плоских задач в упругой динамике. Даются эти решения в виде аналитических функций; на основе теории аналитических функций определяются функции решений из граничных условий. Это значит, что можно использовать методы, разработаны Мусхелишвилим для задач статики теории упругости, для расчета других задач в пределах класса решений подобия. Доказывается что метод Смирнова—Соболева является особим случаем общих результатов, полученных в работе. С целью иллюстрации применения общих результатов, решаются подробно: краевые задачи полуплоскости в напряженном состоянии, под влиянием внезапно приложенной линейной нагрузки, нагрузка растяжения и нагрузка по половине граничной поверхности.